

Ergodic Theory - Week 2

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1 Von Neumann's Mean Ergodic Theorem

P1. Let $(X, \mathcal{A}, \mu, T)$ be a measure-preserving system.

(a) Let $A, B \in \mathcal{A}$. Show that if $(X, \mathcal{A}, \mu, T)$ is ergodic then

$$\mu(A) - \sqrt{\mu(A)(1 - \mu(B))} \leq \limsup_{n \rightarrow \infty} \mu(T^{-n}A \cap B) \quad \text{and} \quad \liminf_{n \rightarrow \infty} \mu(T^{-n}A \cap B) \leq \sqrt{\mu(A)\mu(B)}.$$

By the mean ergodic theorem, we have that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mu(T^{-n}A \cap B) = \mu(A)\mu(B).$$

On the other hand, we have that

$$\mu(A) - \sqrt{\mu(A)(1 - \mu(B))} \leq \mu(A)\mu(B) \leq \sqrt{\mu(A)\mu(B)},$$

given that $\mu(A), \mu(B) \in [0, 1]$ and for $x \in [0, 1]$, $x \leq \sqrt{x}$. Hence, we have

$$\mu(A) - \sqrt{\mu(A)(1 - \mu(B))} \leq \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mu(T^{-n}A \cap B) \leq \sqrt{\mu(A)\mu(B)},$$

from where the inequalities follow.

(b) Let $f \in L^2(X, \mathcal{A}, \mu)$. Show that the limit

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \int T^n f \cdot \bar{f} d\mu$$

exists, is real, and is greater than or equal to $|\int f d\mu|^2$.

By the mean ergodic theorem, we have that $\frac{1}{N} \sum_{n=1}^N T^n f \rightarrow f_{inv}$ as $N \rightarrow \infty$, in $L^2(\mu)$. Therefore, using the dominated convergence theorem to swap the limit and the integral, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \int T^n f \cdot \bar{f} d\mu = \lim_{N \rightarrow \infty} \int \frac{1}{N} \sum_{n=0}^{N-1} T^n f \cdot \bar{f} d\mu = \int f_{inv} \cdot \bar{f}.$$

On the other hand, notice that $\bar{f} = \overline{f_{inv}} + \overline{f_{erg}}$. So, we have that

$$\int f_{inv} \cdot \bar{f} d\mu = \int |f_{inv}|^2 d\mu + \int f_{inv} \cdot \overline{f_{erg}} d\mu = \int |f_{inv}|^2 d\mu.$$

Here, we used the fact that $\overline{f_{erg}}$ is orthogonal to f_{inv} . We conclude that the limit exists and is real and equal to $\int |f_{inv}|^2 d\mu$. All that is left is to prove that is greater than or equal to $|\int f d\mu|^2$. Since 1 is invariant, we have that

$$0 = \langle 1, f_{erg} \rangle = \int f_{erg} d\mu.$$

Thus, the Cauchy-Schwarz inequality yields

$$\left| \int f d\mu \right|^2 = \left(\int f_{inv} d\mu \right)^2 \leq \int |f_{inv}|^2 d\mu.$$

P2. (a) Show that the circle rotation system $(\mathbb{T}, \mathbb{B}_{\mathbb{T}}, m, R_{\alpha})$ is ergodic if and only if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

If $\alpha \in \mathbb{Q}$, then let $n \in \mathbb{N}$ be such that $n\alpha \in \mathbb{Z}$. Then, define $f(t) = e^{2\pi i n t}$ and notice that $R_{\alpha}f = f$ but f is not constant. Hence, the system is not ergodic.

Assume now that α irrational and let $f \in L^2(\mathbb{T})$ be a R_{α} -invariant function. We write its Fourier expansion

$$f(t) = \sum_{n \in \mathbb{Z}} c_n e(nt).$$

Here, the equality and convergence is understood to be in $L^2(\mathbb{T})$. The fact that f is R_{α} invariant yields

$$\sum_{n \in \mathbb{Z}} c_n e(n\alpha) e(nt) = \sum_{n \in \mathbb{Z}} c_n e(nt).$$

Given the uniqueness of the Fourier coefficients, we conclude that $c_n e(n\alpha) = c_n$. Furthermore, we have $e(n\alpha) \neq 1$ for all $n \in \mathbb{N}$, since α is irrational. This implies that for all $n \neq 0$ we have $c_n = 0$. Thus, $f = c_0$ almost everywhere. We conclude that the system is ergodic.

(b) Show that the circle doubling system $(\mathbb{T}, \mathbb{B}_{\mathbb{T}}, m, T_2)$, where $T_2(x) = 2x \pmod{1}$, is ergodic.

More generally, we will show that the map $T_p x = px \pmod{1}$ is ergodic for any $p \in \mathbb{N}$ with $p > 1$. Let $f \in L^2(\mathbb{T})$ be a T_p -invariant function. Again, writing its Fourier expansion

$$f(t) = \sum_{n \in \mathbb{Z}} c_n e(nt),$$

and using the fact that $f \circ T_p = f$ we get that

$$\sum_{n \in \mathbb{Z}} c_n e(pnt) = \sum_{n \in \mathbb{Z}} c_n e(nt).$$

By the uniqueness of the Fourier coefficients, we get that

$$c_n = \begin{cases} c_{n/p}, & \text{if } n \in p\mathbb{Z} \\ 0, & \text{otherwise.} \end{cases}$$

In particular, for every $n \in \mathbb{Z}$, $c_n = c_{pn}$. We prove that $c_n = 0$ for all $n \neq 0$. Indeed, we

recall Parseval's identity:

$$\|f\|_{L^2(\mathbb{T})}^2 = \sum_{n \in \mathbb{Z}} |c_n|^2,$$

which implies that the series on the right-hand side is convergent. Thus, c_n becomes arbitrarily small contradicting our previous equalities. To be more precise, if there is $n_0 \in \mathbb{Z} \setminus \{0\}$ such that $c_{n_0} \neq 0$, then

$$\|f\|_{L^2(\mathbb{T})}^2 = \sum_{m \in \mathbb{Z}} |c_m|^2 \geq \sum_{m \in \mathbb{Z}} |c_{p^m n_0}|^2 = \sum_{m \in \mathbb{Z}} |c_{n_0}|^2 = +\infty,$$

which is a contradiction.

We conclude that $c_n = 0$ for all $n \neq 0$ and then $f = c_0$ almost everywhere. This implies that the system is ergodic.

- P3.** Let $(X, \mathcal{A}, \mu, T)$ be a measure-preserving system. We call $(X, \mathcal{A}, \mu, T)$ **mixing** if for all $A, B \in \mathcal{A}$, we have $\lim_{N \rightarrow \infty} \mu(T^{-N}A \cap B) = \mu(A)\mu(B)$. Show that $(X, \mathcal{A}, \mu, T)$ is mixing if and only if for all $A \in \mathcal{A}$ we have

$$\lim_{N \rightarrow \infty} \mu(T^{-N}A \cap A) = \mu(A)^2. \quad (1)$$

Solution: If the system is mixing, then the statement of (1) follows trivially. For the non-trivial implication, take $A \in \mathcal{A}$. Instead of working with measurable sets, we will prove that for all $f \in L^2(\mu)$ we have that

$$\lim_{n \rightarrow \infty} \int_X T^n \mathbb{1}_A \cdot f d\mu = \mu(A) \cdot \int f d\mu. \quad (2)$$

Indeed, assuming we have proven (2), then plugging in $f = \mathbb{1}_B$ yields our claim.

This statement is trivially true when f is constant and it is also true when $f = T^k \mathbb{1}_A$, since

$$\int T^n \mathbb{1}_A \cdot T^k \mathbb{1}_A d\mu = \int T^{n-k} \mathbb{1}_A \cdot \mathbb{1}_A d\mu = \mu(A \cap T^{n-k}A) \xrightarrow{n \rightarrow +\infty} \mu(A)^2$$

by our hypothesis. This suggests to consider the closed sub-family $\mathcal{H}$ of $L^2(\mu)$ generated by linear combinations of functions in the sets $\{c : c \in \mathbb{R}\}$ and $\{T^k \mathbb{1}_A \mid k \in \mathbb{N}_0\}$. More precisely, we let

$$\mathcal{H} := \overline{\langle \{c : c \in \mathbb{R}\} \cup \{T^k \mathbb{1}_A \mid k \in \mathbb{N}_0\} \rangle}.$$

Here, we are working in the Hilbert space $L^2(\mu)$ with the topology generated by the L^2 -norm. Therefore, when we consider the closure of the set above, we refer to limit points of functions in the L^2 sense.

The statement is valid in the family $\mathcal{H}$. Firstly, the statement is valid in the family that generates $\mathcal{H}$, and by linearity, it is true for all finite linear combinations of functions in the two generating sets. We need to prove that the statement is closed under taking limits. Let $(f_n)_n \subseteq L^2(\mu)$ be a sequence such that $f_n \rightarrow f$ in $L^2(\mu)$ and such that, for all $n \in \mathbb{N}$, f_n

satisfies (2). Then,

$$\begin{aligned}
& \left| \langle \mathbb{1}_A \circ T^n, f \rangle - \mu(A) \int f d\mu \right| \\
& \leq |\langle \mathbb{1}_A \circ T^n, f - f_m \rangle| + \left| \langle \mathbb{1}_A \circ T^n, f_m \rangle - \mu(A) \int f_m d\mu \right| + \left| \mu(A) \int f_m d\mu - \mu(A) \int f d\mu \right| \\
& \leq \|\mathbb{1}_A \circ T^n\|_2 \cdot \|f - f_m\|_2 + \left| \langle \mathbb{1}_A \circ T^n, f_m \rangle - \mu(A) \int f_m d\mu \right| + \mu(A) \|f_m - f\|_2 \\
& = (1 + \mu(A)) \|f_m - f\|_2 + \left| \langle \mathbb{1}_A \circ T^n, f_m \rangle - \mu(A) \int f_m d\mu \right|.
\end{aligned}$$

Taking $\limsup_{N \rightarrow \infty}$, the second term converges to zero because f_m satisfies (2) and, then, if we take $\lim_{M \rightarrow \infty}$ we get that the first term converges to zero by the assumption that f_m converges to f in $L^2(\mu)$. We infer that

$$\limsup_{N \rightarrow \infty} \left| \langle \mathbb{1}_A \circ T^n, f \rangle - \mu(A) \int f d\mu \right| \leq 0,$$

which implies that

$$\lim_{N \rightarrow \infty} \left| \langle \mathbb{1}_A \circ T^n, f \rangle - \mu(A) \int f d\mu \right| = 0.$$

We have proved that all functions in $\mathcal{H}$ satisfy the statement (2). Now, as $\mathcal{H}$ is a closed subspace of the Hilbert space $L^2(\mu)$, we have that $L^2(\mu) = \mathcal{H} \oplus \mathcal{H}^\perp$. Thus, for $f \in L^2(\mu)$ we have the decomposition $f = f_{\mathcal{H}} + f_{\mathcal{H}^\perp}$ where $f_{\mathcal{H}}$ is the projection of f on $\mathcal{H}$. Notice that $1 \in \mathcal{H}$, which implies that $0 = \langle 1, f_{\mathcal{H}^\perp} \rangle = \int f_{\mathcal{H}^\perp} d\mu$, and therefore

$$\int f d\mu = \int f_{\mathcal{H}} d\mu.$$

Now, we compute the limit:

$$\begin{aligned}
\lim_{n \rightarrow \infty} \langle \mathbb{1}_A \circ T^n, f \rangle &= \lim_{n \rightarrow \infty} \langle \mathbb{1}_A \circ T^n, f_{\mathcal{H}} \rangle + \langle \mathbb{1}_A \circ T^n, f_{\mathcal{H}^\perp} \rangle \\
&= \lim_{n \rightarrow \infty} \langle \mathbb{1}_A \circ T^n, f_{\mathcal{H}} \rangle + 0 \\
&= \mu(A) \int f_{\mathcal{H}} d\mu = \mu(A) \int f d\mu,
\end{aligned}$$

and the conclusion follows.

P4. Let X be a compact metric space, and let $T : X \rightarrow X$ be continuous. Suppose that μ is a T -invariant ergodic probability measure defined on the Borel subsets of X . Prove the following:

(a) The support of the measure μ , defined as

$$\text{supp}(\mu) = X \setminus \left(\bigcup_{\substack{U \subseteq X \text{ open} \\ \mu(U)=0}} U \right),$$

has full measure.

We prove this in the more general setting where (X, d) is a separable metric space. Given a dense sequence x_n on X , the sequence of open balls $(B(x_n, \frac{1}{m}))_{m, n \in \mathbb{N}}$ forms a countable base for the topology on (X, d) (meaning every open set contains an open ball from our

countable collection). This is an easy consequence of the triangle inequality: if U is open, there exists $v \in U, m \in \mathbb{N}$ such that $B(v, \frac{1}{2m}) \subset U$ and we can also find x_n such that $d(x_n, v) < \frac{1}{2m}$ from which we derive $B(x_n, \frac{1}{2m}) \subset U$.

We denote $(V_n)_n$ this countable base of X for convenience. Then notice that for every $U \subseteq X$ open with $\mu(U) = 0$, we can write

$$U = \bigcup_{n \in \mathbb{N}, V_n \subseteq U} V_n.$$

Observe that $V_n \subseteq U$ implies $\mu(V_n) = 0$. Therefore, we conclude that

$$\left(\bigcup_{\substack{U \subseteq X \text{ open} \\ \mu(U)=0}} U \right) = \left(\bigcup_{\substack{n \in \mathbb{N} \\ \mu(V_n)=0}} V_n \right).$$

As the right-hand side of the equation is a countable union of sets of measure 0, we conclude that this set has measure 0. It follows that $\mu(\text{supp}(\mu)) = 1$.

- (b) For μ -almost every $x \in X$ and for every $y \in \text{supp}(\mu)$, there exists a sequence $n_k \nearrow \infty$ such that $T^{n_k}x \rightarrow y$ as $k \rightarrow \infty$.

Solution: We want to prove that

$$A = \{x \in X : \text{for all } y \in \text{supp}(\mu) \text{ there exists } n_k \rightarrow +\infty \text{ such that } T^{n_k}x \rightarrow y\}$$

has full measure. We would like to take countable many $y \in \text{supp}(\mu)$, in order to reduce the statement to an easier one. For this we note that $\text{supp}(\mu)$ is compact by being a closed subset of a compact set. Thus, $\text{supp}(\mu)$ is separable and we can take $(y_n)_n \subseteq \text{supp}(\mu)$ to a dense sequence in Y . Define

$$B = \{x \in X : \text{for all } m \in \mathbb{N} \text{ there exists } n_k \rightarrow +\infty \text{ such that } T^{n_k}x \rightarrow y_m\}.$$

Clearly $A \subseteq B$. We will prove that $A = B$. For the reverse inclusion, let $x \in B$, $y \in \text{supp}(\mu)$, and $\varepsilon > 0$. By density, there exists $m \in \mathbb{N}$, such that $d(y_m, y) < \varepsilon$. Since $y_m \in \text{supp}(\mu)$, there exists a sequence $n_k \rightarrow +\infty$, such that $T^{n_k}x \rightarrow y_m$. Thus, for all k sufficiently large, we have $d(T^{n_k}x, y_m) < \varepsilon/2$. We infer that

$$d(T^{n_k}x, y) \leq d(T^{n_k}x, y_m) + d(y_m, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon.$$

Therefore, for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $d(T^n x, y) < \varepsilon$ which readily implies that $y \in \overline{\{T^n x : n \in \mathbb{N}\}}$. We conclude that $x \in A$ and thus $B \subseteq A$, as we claimed.

It suffices to show that B has full measure. Notice that

$$X \setminus B \subseteq \bigcup_{m \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \left(X \setminus T^{-n} B(y_m, \frac{1}{k}) \right).$$

Since all unions are countable, it suffices to show that

$$\mu \left(\bigcap_{n \in \mathbb{N}} X \setminus T^{-n} B \left(y_m, \frac{1}{k} \right) \right) = 0 \iff \mu \left(\bigcup_{n \in \mathbb{N}} T^{-n} B \left(y_m, \frac{1}{k} \right) \right) = 1.$$

Since $y_m \in \text{supp}(\mu)$, we have that $\mu(B(y_m, \frac{1}{k})) > 0$. Observe that the set $\bigcup_{n \in \mathbb{N}} T^{-n} B(y_m, \frac{1}{k})$ is T -invariant and due to ergodicity it has measure zero or 1. However, it cannot have measure 0, since $\mu(B(y_m, \frac{1}{k})) > 0$. The conclusion follows.